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A Linear Connection in a
Principal Bundle

by

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DEPARTMENT OF MATHEMATICS

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1. Introduction

Initially the main aim of this paper was to investigate the
relationship of the dynamics of rigid bodies with the
the group of rigid transformations. This is done in
section 2. The group of rigid transformations is defined as the
set of all transformations which preserve the distance
between any two points. This group is isomorphic to the
direct product of the orthogonal group and the translation
group. The orthogonal group is the group of all linear
transformations which preserve the inner product. The
translation group is the group of all translations.

The group of rigid transformations is a Lie group. The
Lie algebra of this group is the direct sum of the
Lie algebra of the orthogonal group and the Lie algebra
of the translation group. The Lie algebra of the
orthogonal group is the space of all skew-symmetric
matrices. The Lie algebra of the translation group is
the space of all vectors. The Lie algebra of the
group of rigid transformations is the direct sum of
these two spaces.

The group of rigid transformations is a subgroup of the
group of all transformations. The group of all
transformations is the group of all linear
transformations. The group of rigid transformations
is a subgroup of this group.

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0. Introduction.

Initially the main aim of this paper was to generalize Arnold's treatment of the dynamics of rigid bodies (see [1]). In the latter, a Lie group G with a left-invariant metric is considered. The movements of the generalized rigid bodies are then (when no exterior forces are present) the geodesics for the metric. The variational equations for the geodesics give the classical results first derived by Euler in 1765 for $G = SO(3)$ (viz., the equations of motion and the conservation laws).

The presence of a connection on G can be used to obtain the variational equations in a slightly different way. One may namely apply the Euler derivative introduced by Peetre in [2], in particular the form it takes when a connection on the manifold under consideration is taken into account [3]. For the problem at hand, the natural choice of connection on G is not the Levi-Civita, but rather the flat connection defined by

$$\omega(\nabla_X Y) = X(\omega(Y)),$$

where ω is the left-invariant Maurer-Cartan form on G . (Note that Y is parallel precisely when $\omega(Y)$ is constant.)

Seeking a generalization to the curved case, we consider a principal bundle P over a manifold M equipped with an affine connection. A suitable linear connection on P is defined and characterized. Then the connection is described in local coordinates. Thereafter the torsion and curvature tensors are determined. These results are of interest in their own right and comprise the bulk of the paper, though in concluding we of course also generalize Arnold's result.

1. Introduction of a linear connection in a principal bundle.

Let $P = P(M, G)$ be a principle bundle over a manifold M with structure group G and canonical projection $\pi: P \rightarrow M$ ($M = P/G$). Assume that a connection on P is given by the form

$$\omega: TP \rightarrow \underline{g},$$

where $\underline{g}$ is the Lie algebra of G , and assume furthermore that M comes equipped with a linear connection ∇ .

Making use of these two connections we define a linear connection $\bar{\nabla}$ on P . If $p \in P$, X and Y are vector fields on P , and t is the parameter along the integral curve of X through p , with $t = 0$ corresponding to the point p , define

$$\bar{\nabla}_X Y = \left(\frac{\nabla}{dt} \pi_* Y_t \Big|_{t=0} \right)^h + ((X(\omega(Y)))_p)^*,$$

which we for the sake of brevity write as

$$(1.1) \quad \bar{\nabla}_X Y = \left(\frac{\nabla}{dt} \pi_* Y_t \right)^h + (X(\omega(Y)))^*.$$

Here h denotes horizontal lift (of tangent vectors on M to tangent vectors on P) and A^* the fundamental vector field (on P) associated with $A \in \underline{g}$.

We want to show that $\bar{\nabla}$ has the properties of a Koszul connection. Linearity over $C^\infty(P)$ in X is obvious, as is linearity over $\mathbb{R}$ in Y . Thus there remains to show that

$$\bar{\nabla}_X(fY) = (Xf)Y + f \bar{\nabla}_X Y$$

for $f \in C^\infty(P)$.

We start with the simple equalities

$$\pi_*(fY)_p = f(p) \pi_* Y_p$$

and

$$\omega(fY) = f\omega(Y).$$

Combining these with

$$(\pi_* Y)^h = hY \quad \text{and} \quad \omega(Y)^* = vY \quad (\text{pointwise}),$$

where hY and vY are the horizontal and vertical components of Y , we obtain

$$\begin{aligned} \bar{\nabla}_X(fY) &= ((Xf) \pi_* Y + f \frac{\nabla}{dt} \pi_* Y)^h + ((Xf) \omega(Y) + f X(\omega(Y)))^* = \\ &= (Xf)(hY + vY) + f \left[\left(\frac{\nabla}{dt} \pi_* Y \right)^h + (X(\omega(Y)))^* \right] = \\ &= (Xf)Y + f \bar{\nabla}_X Y, \end{aligned}$$

which is the desired conclusion.

Returning to the definition, note that

$$\bar{\nabla}_X Y = 0 \iff \begin{cases} \frac{\nabla}{dt} \pi_* Y_t = 0 \\ \frac{d}{dt} \omega(Y_t) = 0, \end{cases}$$

i.e., Y is parallel if and only if its projection is parallel and $\omega(Y)$ is constant. This leads to the following axiomatic characterization of $\bar{\nabla}$:

Proposition 1.1. Assume that P and M are as above. Then the above connection $\bar{\nabla}$ is the unique linear connection on P which satisfies

$$\begin{cases} \pi_* \bar{\nabla} = \nabla \pi_* \\ \bar{\nabla} \omega = 0 \end{cases}.$$

Proof:

Since π_* annihilates vertical vectors and

$$\pi_*(Z^h) = Z,$$

the first equality follows upon applying π_* to both sides of (1.1).

On the other hand, ω annihilates horizontal vectors and

$$\omega(A^*) = A.$$

This gives the second equality in

$$\bar{\nabla}_X \omega(Y) = X(\omega(Y)) - \omega(\bar{\nabla}_X Y) = X(\omega(Y)) - X(\omega(Y)) = 0,$$

and therefore $\bar{\nabla} \omega = 0$ as desired. Since uniqueness is clear now, the proof is complete.

2. The connection $\bar{\nabla}$ in local coordinates.

In this section we describe the connection $\bar{\nabla}$, defined in §1, in local coordinates. First, local coordinates are obtained in the natural manner from those on M and G . Next we derive expressions in these local coordinates for the $*$ operation and the form ω . Then the Christoffel symbols are computed, and the local connection form written out.

Let p_0 be a given point in P . We take local coordinates

$$x^1, \dots, x^n$$

in a neighborhood U of $\pi(p_0) = x_0$ in M . For a given basis

$$E_1, \dots, E_m$$

in $\underline{g}$ we use normal local coordinates near the neutral element e in G , i.e.,

$$\exp(y^\alpha E_\alpha) \mapsto (y^1, \dots, y^m).$$

(Here and in the sequel we use the summation convention, and Greek letters when referring to the y indices.) If σ is a local section of P over U with say

$$\sigma(x_0) = p_0,$$

we can define local coordinates near p_0 by the mapping

$$p = \sigma(x) \exp Y \mapsto (x^1, \dots, x^n, y^1, \dots, y^m),$$

where $x = (x^1, \dots, x^n)$ and $Y = y^\alpha E_\alpha$.

Next we seek an expression in local coordinates for the connection form ω . We begin by considering the left-invariant Maurer-Cartan form, which is a mapping $\Lambda: TG \rightarrow \underline{g}$. The composition with

$$T\underline{g} = \underline{g} \oplus \underline{g} \ni (Y, Z) \mapsto d \exp_{\exp Y}(Z) \in TG,$$

thus

$$T\underline{g} \xrightarrow{\exp_*} TG \xrightarrow{\Lambda} \underline{g},$$

will also be denoted by Λ and is a $\text{hom}(\underline{g}, \underline{g})$ -valued function of y ,

$$\Lambda = \Lambda_\beta^\alpha(y)$$

with respect to the basis $E_1, \dots, E_m$.

By means of the section σ used to define local coordinates we get a local trivialization of P . This local isomorphism of P with $U \times G$ allows us to pull back Λ with the projection $\pi_1: U \times G \rightarrow G$.

Setting $\pi_1^* \Lambda = \tilde{\Lambda}$, we can according to the above interpret $\tilde{\Lambda}$ as a function on $U \times G$ with values in $\text{hom}(\underline{g}, \underline{g})$. The fundamental vector fields $E_1^*, \dots, E_m^*$ on P induced by the action of G can then be written locally as

$$(2.1) \quad E_\beta^* = (\tilde{\Lambda}^{-1})_\beta^\alpha \frac{\partial}{\partial y^\alpha}.$$

For the connection form ω there holds on $U \times G$ the equality

$$\omega = \text{Ad}(\exp Y)^{-1} \pi^* \sigma^* \omega + \tilde{\Lambda}.$$

Set $\text{Ad}(\exp Y)^{-1} = C(Y) \in \text{hom}(\underline{g}, \underline{g})$. Now locally TM is equal to $TV = V \oplus V$, $V = \mathbb{R}^n$, and TP to $T\bar{V}$, $\bar{V} = V \oplus \underline{g}$. The composition

$$\bar{V} \oplus V \rightarrow TV \rightarrow TM \xrightarrow{\sigma^*} TP \xrightarrow{\omega} \underline{g},$$

where the first mapping is the obvious projection and the second the parametrization, will be denoted by $\tilde{\omega}$, and $\tilde{\omega}(x, y) \in \text{hom}(V, \underline{g})$. Thus we may write

$$(2.2) \quad \omega = E_\alpha (C_\beta^{\alpha \tilde{\omega}} dx^i + \tilde{\Lambda}_\beta^\alpha dy^\beta).$$

To obtain an expression for the local connection form $\bar{\theta}$ for the linear connection $\bar{\nabla}$, we first compute the Christoffel symbols. We start by examining

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \left(\frac{\nabla}{dt} \pi_* \frac{\partial}{\partial x^j} \right)^h + \left(\frac{\partial}{\partial x^i} \left(\omega \left(\frac{\partial}{\partial x^j} \right) \right) \right)^*.$$

If we (with the hope that the context will make things clear) denote by $\frac{\partial}{\partial x^k}$ the locally defined vector fields on M and on P induced by the coordinates x , we see that

$$\frac{\nabla}{dt} \pi_* \frac{\partial}{\partial x^j} = \frac{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}}{\frac{\partial x^i}{\partial x^i}} = \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

where Γ_{ij}^k are the Christoffel symbols of ∇ . The horizontal lift of $\frac{\partial}{\partial x^k}$ is

$$\begin{aligned} \left(\frac{\partial}{\partial x^k} \right)^h &= \frac{\partial}{\partial x^k} - \omega \left(\frac{\partial}{\partial x^k} \right)^* = \frac{\partial}{\partial x^k} - (E_\beta C_\gamma^{\beta \tilde{\omega}})^* = \\ &= \frac{\partial}{\partial x^k} - (\tilde{\Lambda}^{-1})_\beta^\alpha (C \tilde{\omega})_k^\beta \frac{\partial}{\partial y^\alpha}, \end{aligned}$$

which follows from (2.1) and (2.2).

We also need the fact that

$$\left(\frac{\partial}{\partial x^i} \left(\omega \left(\frac{\partial}{\partial x^j} \right) \right) \right)^* = (E_\beta \frac{\partial}{\partial x^i} (C \tilde{\omega})_j^\beta)^* = (\tilde{\Lambda}^{-1})_\beta^\alpha \frac{\partial}{\partial x^i} (C \tilde{\omega})_j^\beta \frac{\partial}{\partial y^\alpha}.$$

This combined with the previous paragraph yields

$$\bar{\Gamma}_{ij}^k = \pi^* \Gamma_{ij}^k$$

and

$$\bar{\Gamma}_{ij}^\alpha = (\tilde{\Lambda}^{-1})^\alpha_\beta \left\{ \frac{\partial}{\partial x^i} (C\tilde{\omega})^\beta_j - (C\tilde{\omega})^\beta_k \pi^* \Gamma_{ij}^k \right\},$$

with the obvious notation for the Christoffel symbols of $\bar{\nabla}$.

Proceeding in a similar fashion, one obtains

Proposition 2.1. The Christoffel symbols for $\bar{\nabla}$ in the above local coordinates are

$$\bar{\Gamma}_{ij}^k = \pi^* \Gamma_{ij}^k, \quad \bar{\Gamma}_{ij}^\gamma = (\tilde{\Lambda}^{-1})^\gamma_\beta \left\{ \frac{\partial}{\partial x^i} (C\tilde{\omega})^\beta_j - (C\tilde{\omega})^\beta_k \pi^* \Gamma_{ij}^k \right\},$$

$$\bar{\Gamma}_{i\alpha}^k = 0, \quad \bar{\Gamma}_{i\alpha}^\gamma = 0,$$

$$\bar{\Gamma}_{\alpha i}^k = 0, \quad \bar{\Gamma}_{\alpha i}^\gamma = (\tilde{\Lambda}^{-1})^\gamma_\beta \frac{\partial}{\partial y^\alpha} (C\tilde{\omega})^\beta_i,$$

$$\bar{\Gamma}_{\alpha\beta}^k = 0, \text{ and } \bar{\Gamma}_{\alpha\beta}^\gamma = (\tilde{\Lambda}^{-1})^\gamma_\delta \frac{\partial}{\partial y^\alpha} \tilde{\Lambda}^\delta_\beta.$$

The local identification of TM with TV allows us to consider the local connection form θ belonging to the linear connection ∇ on M as a $\text{hom}(V, V) \otimes V^*$ -valued function of x , where

$$\theta_j^k = \Gamma_{ij}^k dx^i.$$

Since $\pi^* dx^i = dx^i$, $(\pi^* \theta)_j^k = \pi^* \Gamma_{ij}^k dx^i$.

In the same way the local connection form $\bar{\theta}$ associated with $\bar{\nabla}$ is $\text{hom}(\bar{V}, \bar{V}) \otimes \bar{V}^*$ -valued. In matrix notation

$$\text{hom}(\bar{V}, \bar{V}) = \begin{pmatrix} \text{hom}(V, V) & \text{hom}(\underline{g}, V) \\ \text{hom}(V, \underline{g}) & \text{hom}(\underline{g}, \underline{g}) \end{pmatrix}.$$

By definition

$$\bar{\theta}_j^k = \bar{\Gamma}_{ij}^k dx^i + \bar{\Gamma}_{\alpha j}^k dy^\alpha,$$

$$\bar{\theta}_\beta^k = \bar{\Gamma}_{i\beta}^k dx^i + \bar{\Gamma}_{\alpha\beta}^k dy^\alpha,$$

$$\bar{\theta}_j^\gamma = \bar{\Gamma}_{ij}^\gamma dx^i + \bar{\Gamma}_{\alpha j}^\gamma dy^\alpha,$$

and
$$\bar{\theta}_\beta^\gamma = \bar{\Gamma}_{i\beta}^\gamma dx^i + \bar{\Gamma}_{\alpha\beta}^\gamma dy^\alpha.$$

Recall that $C \in \text{hom}(\underline{g}, \underline{g})$, $\tilde{\omega} \in \text{hom}(V, \underline{g})$, and $\tilde{\Lambda} \in \text{hom}(\underline{g}, \underline{g})$. Then Proposition 2.1 and the remarks that followed give

Proposition 2.2. Using the above notations,

$$\bar{\theta} = \begin{pmatrix} I & 0 \\ 0 & \tilde{\Lambda}^{-1} \end{pmatrix} \begin{pmatrix} \pi^*\theta & 0 \\ d(C \cdot \tilde{\omega}) - C \cdot \tilde{\omega} \cdot \pi^*\theta & d\tilde{\Lambda} \end{pmatrix}.$$

Remark: Note that by the calculations leading to Proposition 2.1

$$\begin{pmatrix} 0 \\ \tilde{\Lambda}^{-1} \end{pmatrix} \in \text{hom}(\underline{g}, \bar{V})$$

corresponds to the $*$ operation, and

$$\begin{pmatrix} I \\ -\tilde{\Lambda}^{-1} \cdot C \cdot \tilde{\omega} \end{pmatrix} \in \text{hom}(V, \bar{V})$$

corresponds to h . Therefore we may write Proposition 2.2 as

$$\bar{\theta} = \tilde{\theta}^h + (d\omega)^*,$$

where $\tilde{\theta} = (\pi^*\theta \quad 0) \in \text{hom}(\bar{V}, V) \otimes \bar{V}^*$ and
 $\omega = (C\tilde{\omega} \quad \tilde{\Lambda}) \in \text{hom}(\bar{V}, \underline{g})$.

3. The torsion and curvature tensors.

Using the description of the linear connection $\bar{\nabla}$ in local coordinates obtained in the previous section, it is easy to compute the torsion and curvature tensors. Nonetheless, we relegate this computation to the appendix and give here a direct and perhaps more satisfying proof which does not use local coordinates.

Recall that for vector fields X, Y and Z on P , we have

$$\bar{T}(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

and

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z$$

for the torsion $\bar{T}$ and curvature $\bar{R}$ of $\bar{\nabla}$.

Proposition 3.1. For the torsion tensor $\bar{T}$ and curvature tensor $\bar{R}$ of the linear connection $\bar{\nabla}$ on P , there hold the equalities

$$(3.1) \quad \bar{T}(X, Y) = T(\pi_* X, \pi_* Y)^h + 2d\omega(X, Y)^*$$

and

$$(3.2) \quad \bar{R}(X, Y)Z = (R(\pi_* X, \pi_* Y)\pi_* Z)^h,$$

where T and R are the torsion and curvature of the connection ∇ on M .

Proof:

We start with (3.1). Note first that for vector fields X and Y on P we have pointwise

$$\begin{aligned} [X, Y] &= h[X, Y] + \omega([X, Y])^* = \\ &= (\pi_*[X, Y])^h + \omega([X, Y])^*. \end{aligned}$$

Thus

$$\begin{aligned} \bar{T}(X, Y) &= \left(\frac{\nabla}{dt} \pi_* Y_t - \frac{\nabla}{ds} \pi_* X_s - \pi_*[X, Y] \right)^h + \\ &\quad + \{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\}^*, \end{aligned}$$

t and s being the parameters along the integral curves of X and Y respectively. Since

$$2d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]),$$

this explains the presence of the second term on the right-hand side of (3.1).

To motivate the first term, we note first that $\bar{T}$ is a tensor. At a given point p , we need therefore only consider the following three cases:

- 1) X_p, Y_p are horizontal
- 2) X_p horizontal, Y_p vertical
- 3) X_p, Y_p both vertical

(and both vectors $\neq 0$).

In the first case we extend x_p and y_p to commuting vector fields tangential to a two-dimensional submanifold of P through p diffeomorphic with its projection on M . Then $[X, Y] = 0$, and we must have

$$\frac{\nabla}{dt} \pi_* Y_t - \frac{\nabla}{ds} \pi_* X_s - \pi_* [X, Y] = T(\pi_* X, \pi_* Y).$$

In the last two cases, $\pi_* Y_p = 0$ and therefore $T(\pi_* X_p, \pi_* Y_p) = 0$. Consequently we want to extend X_p and Y_p appropriately so as to obtain

$$(*) \quad \frac{\nabla}{dt} \pi_* Y_t - \frac{\nabla}{ds} \pi_* X_s - \pi_* [X, Y] = 0.$$

In case 2), we extend X_p to a horizontal vector field invariant by R_a , $a \in G$, and Y_p to $\omega(Y_p)^*$. Then the definition of the Lie derivative shows that $[Y, X] = 0$. Furthermore, $\pi_* Y_t = 0$ and $\pi_* X_s$ is constant since X is invariant by R_a . This gives (*).

In the final case, (*) is obtained by extending X_p and Y_p to vertical vector fields X and Y respectively both invariant by R_a , $a \in G$. This completes the proof of (3.1).

The proof of (3.2) is similar. For vector fields X, Y , and Z , we have

$$\begin{aligned}
 (+) \quad \bar{R}(X, Y)Z &= \left\{ \frac{\nabla}{dt} \left(\frac{\nabla}{ds} \pi_* Z_s \right)_t - \frac{\nabla}{ds} \left(\frac{\nabla}{dt} \pi_* Z_t \right)_s - \frac{\nabla}{dr} \pi_* Z_r \right\}^h + \\
 &\quad + \{ (XY - YX - [X, Y]) (\omega(Z)) \}^*,
 \end{aligned}$$

t and s as above, and r denoting the parameter along the integral curves of $[X, Y]$. The second term obviously vanishes, so we need only investigate the first.

If Z_p is vertical, a vertical extension immediately yields $\bar{R}(X, Y)Z = 0$ and the desired conclusion. We assume therefore that Z_p is horizontal, and consider once more the cases 1), 2), and 3) for X_p and Y_p used in the proof of (3.1).

When X_p and Y_p are horizontal we extend them precisely as before to obtain $[X, Y]_p = 0$. Extending Z_p in some way it is clear from (+) that (3.2) is valid in this case.

In the second case, we extend X_p and Z_p to horizontal vector fields invariant by G , and Y_p to $\omega(Y_p)^*$. As before, $[X, Y] = 0$. Since X and Z are G -invariant, so is

$$\left(\frac{\nabla}{dt} \pi_* Z_t\right)^h.$$

Thus, Y being vertical, we obtain

$$\frac{\nabla}{ds} \pi_* Z_s = 0 \quad \text{and} \quad \frac{\nabla}{ds} \left(\frac{\nabla}{dt} \pi_* Z_t\right)_s = 0.$$

As a result, $\bar{R}(X, Y)Z = 0$.

Case 3) is resolved by extending X_p and Y_p to $\omega(X_p)^*$ and $\omega(Y_p)^*$ respectively, and Z_p to a G -invariant horizontal field. Then

$$\frac{\nabla}{dt} \pi_* Z_t = 0, \quad \frac{\nabla}{ds} \pi_* Z_s = 0, \quad \text{and} \quad \frac{\nabla}{dr} \pi_* Z_r = 0,$$

because $\pi_* Z$ is constant in each equation. In the final equation this follows from the properties of fundamental vector fields, in particular that

$$\omega([X, Y]_p)^* = [X, Y]$$

when $X = \omega(X_p)^*$ and $Y = \omega(Y_p)^*$.

4. Geodesic lines.

We now use this connection to obtain a generalization of Arnold's treatment of the dynamics of rigid bodies.

Here we assume that M is pseudo-Riemannian with metric tensor g , and take ∇ to be the Levi-Civita connection. Furthermore, let Q be a non-degenerate quadratic form on $\underline{g}$ ("inertia form"). This allows us to define a metric $\bar{g}$ on P by the formula

$$\bar{g}(X, Y) = g(\pi_* X, \pi_* Y) + Q(\omega(X), \omega(Y)).$$

We note that $\bar{\nabla} \bar{g} = 0$ since $\nabla g = 0$, $\nabla \pi_* = \pi_* \nabla$, and $Z(\omega(X)) = \omega(\bar{\nabla}_X X)$.

Taking our Lagrange function f as

$$f(X) = \frac{1}{2} \bar{g}(X, X)$$

for $X \in TP$, we set ourselves the task of finding the extremals (geodesics). We stress once more that the main interest here lies in the consideration of a connection natural for the problem and different from the Levi-Civita. This can be seen in the example $P = G = SO(3)$ (for motions of a rigid body) and ω = the left-invariant Maurer-Cartan form. Here we want Y parallel if and only if $\omega(Y)$ is constant, and hence $\omega(\bar{\nabla}_X Y) = X(\omega(Y))$.

To obtain the extremals, we use the general expression for the "Euler derivative" (see Peetre, [3])

$$E_Y f(X) = \bar{\nabla}_Y f(X) - (\bar{\nabla}_X \mu)(Y) + \mu(\bar{T}(Y, X)).$$

Here f can be any function on TP and $\mu(Y) = \dot{f}(Y; X)$, $\dot{f}$ being the fiber derivative of f defined by

$$\dot{f}(Y; X) = \left. \frac{d}{dt} f(X + tY) \right|_{t=0}.$$

Using $\bar{\nabla} \bar{g} = 0$ and $\mu(Y) = \bar{g}(Y, X)$, we get in our case

$$E_Y f(X) = -\bar{g}(Y, \bar{\nabla}_X X) + \bar{g}(\bar{T}(Y, X), X).$$

Because ∇ is the Levi-Civita connection on M , $T = 0$. The expression (3.1) for $\bar{T}$ therefore simplifies to

$$\bar{T}(Y, X) = 2d\omega(Y, X)^*.$$

Hence $E_Y f(X) = 0$ is equivalent with

$$(4.1) \quad -g(\pi_* Y, \pi_* (\bar{\nabla}_X X)) - Q(\omega(Y), X(\omega(X))) + \\ + Q(2d\omega(Y, X), \omega(X)) = 0.$$

The structure equation for the connection ω is

$$2d\omega(Y,X) = -[\omega(Y), \omega(X)] + 2\Omega(Y,X),$$

where Ω is the curvature of ω . For horizontal Y , $\omega(Y) = 0$ and

$$d\omega(Y,X) = \Omega(Y,X).$$

If Y instead is vertical, then $\Omega(Y,X) = 0$ since $hY = 0$, and

$$2d\omega(Y,X) = [\omega(X), \omega(Y)] = \text{ad } \omega(X) \cdot \omega(Y).$$

Set $\omega(X) = A$. Choosing Y in (4.1) first horizontal and then vertical, we see therefore that the Euler equations ($E_Y f(X) = 0$) are

$$(4.2) \quad \begin{cases} g(\pi_* \cdot, \pi_*(\bar{\nabla}_X X)) - 2Q(\Omega(\cdot, X), A) = 0 \\ X(A) = (\text{ad } A) * A, \end{cases}$$

where $*$ denotes conjugation with respect to Q . Note that if $\Omega = 0$, the first equation becomes

$$\frac{\nabla}{dt} \pi_* X_t = 0,$$

with t the parameter along the integral curves of X . The equations (4.2) constitute the desired generalization of Arnold's result.

Appendix

We prove Proposition 3.1 by means of Proposition 2.2. We use the notations of sections 2 and 3.

For the connection ∇ on M , the torsion is given locally by

$$T = 2\theta \wedge dx,$$

with

$$(dx) = \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix}$$

Therefore,

$$(A.1) \quad \pi^* T = 2\pi^* \theta \wedge dx$$

by our choice of local coordinates.

In local coordinates the torsion tensor $\bar{T}$ of the linear connection $\bar{\nabla}$ on P satisfies

$$\bar{T} = 2\bar{\theta} \wedge \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where

$$(dy) = \begin{pmatrix} dy^1 \\ \vdots \\ dy^m \end{pmatrix}.$$

Hence, using Proposition 2.2, we obtain

$$(A.2) \quad \begin{aligned} \bar{T} &= 2 \begin{pmatrix} I & 0 \\ 0 & \tilde{\Lambda}^{-1} \end{pmatrix} \begin{pmatrix} \pi^* \theta \wedge dx \\ d(C \cdot \tilde{\omega}) \wedge dx - C \cdot \tilde{\omega} \cdot \pi^* \theta \wedge dx + d\tilde{\Lambda} \wedge dy \end{pmatrix} = \\ &= \begin{pmatrix} I \\ -\tilde{\Lambda}^{-1} \cdot C \cdot \tilde{\omega} \end{pmatrix} 2\pi^* \theta \wedge dx + \begin{pmatrix} 0 \\ \tilde{\Lambda}^{-1} \end{pmatrix} 2(d(C \cdot \tilde{\omega}) \wedge dx + d\tilde{\Lambda} \wedge dy). \end{aligned}$$

Recall that

$$(A.3) \quad \omega = C \cdot \tilde{\omega} \cdot dx + \tilde{\Lambda} \cdot dy$$

in the basis $E_1, \dots, E_m$. Then (A.1), (A.2), and (A.3) combined with the remark at the end of § 2 give

$$\bar{T} = (\pi^* T)^h + (2d\omega)^*,$$

which is precisely (3.1).

Locally the curvature $\bar{R}$ is given by

$$\bar{R} = 2(d\bar{\theta} + \bar{\theta} \wedge \bar{\theta}).$$

We write out $d\bar{\theta}$ and $\bar{\theta} \wedge \bar{\theta}$ separately:

$$d\bar{\theta} = \begin{pmatrix} 0 & 0 \\ 0 & d\tilde{\Lambda}^{-1} \end{pmatrix} \begin{pmatrix} \pi^*\theta & 0 \\ d(C \cdot \tilde{\omega}) - C \cdot \tilde{\omega} \cdot \pi^*\theta & d\tilde{\Lambda} \end{pmatrix} + \\ + \begin{pmatrix} I & 0 \\ 0 & \tilde{\Lambda}^{-1} \end{pmatrix} \begin{pmatrix} \pi^*d\theta & 0 \\ -d(C \cdot \tilde{\omega}) \wedge \pi^*\theta - C \cdot \tilde{\omega} \cdot \pi^*d\theta & 0 \end{pmatrix}$$

and

$$\bar{\theta} \wedge \bar{\theta} = \begin{pmatrix} \pi^*(\theta \wedge \theta) & 0 \\ \left(\tilde{\Lambda}^{-1} \cdot (d(C \cdot \tilde{\omega}) - C \cdot \tilde{\omega} \cdot \pi^*\theta) \wedge \pi^*\theta + \right. & \tilde{\Lambda}^{-1} d\tilde{\Lambda} \cdot \tilde{\Lambda}^{-1} \cdot d\tilde{\Lambda} \\ \left. \tilde{\Lambda}^{-1} d\tilde{\Lambda} \wedge \tilde{\Lambda}^{-1} (d(C \cdot \tilde{\omega}) - C \cdot \tilde{\omega} \cdot \pi^*\theta) \right) & \end{pmatrix}$$

Since $d\tilde{\Lambda}^{-1} = -\tilde{\Lambda}^{-1} \cdot d\tilde{\Lambda} \cdot \tilde{\Lambda}^{-1}$, we obtain

$$\bar{R} = 2 \begin{pmatrix} \pi^*(d\theta + \theta \wedge \theta) & 0 \\ -\tilde{\Lambda}^{-1} \cdot C \cdot \tilde{\omega} \cdot \pi^*(d\theta + \theta \wedge \theta) & 0 \end{pmatrix} = (\pi^*R)^h ,$$

$R = 2(d\theta + \theta \wedge \theta)$ being the curvature of ∇ . This completes the proof.

References.

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